

A NOTE ON THE STABILITY FOR KAWAHARA-KDV TYPE EQUATIONS.

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ABSTRACT. In this paper we establish the nonlinear stability of solitary traveling-wave solutions for the Kawahara-KdV equation

$$u_t + uu_x + u_{xxx} - \gamma_1 u_{xxxxx} = 0,$$

and the modified Kawahara-KdV equation

$$u_t + 3u^2 u_x + u_{xxx} - \gamma_2 u_{xxxxx} = 0,$$

where $\gamma_i \in \mathbb{R}$ is a positive number when $i = 1, 2$. The main approach used to determine the stability of solitary traveling-waves will be the theory developed by Albert in [1].

1. INTRODUCTION.

This work presents the existence of a smooth branch of solitary traveling wave solutions as well as the orbital stability related to Kawahara-Korteweg-de Vries and modified Kawahara-Korteweg-de Vries equations (Kawahara and modified Kawahara equations respectively, henceforth),

$$u_t + uu_x + u_{xxx} - \gamma_1 u_{xxxxx} = 0, \tag{1.1}$$

and

$$u_t + 3u^2 u_x + u_{xxx} - \gamma_2 u_{xxxxx} = 0, \tag{1.2}$$

where $\gamma_i > 0$ when $i = 1, 2$ and $u := u(x, t)$ is a real function. Here, we consider $x \in \mathbb{R}$ and $t \in \mathbb{R}$. These equations model the propagation on nonlinear water-waves in the long-wavelength as in the case KdV's equations. Roughly speaking, such a model-scenario is expected because, if u be a smooth solution of (1.1) and (1.2), then for $\gamma_i \rightarrow 0$ uniformly, $i = 1, 2$ we obtain that u is a solution of the Korteweg-de Vries and modified Korteweg-de Vries equations,

$$u_t + uu_x + u_{xxx} = 0, \tag{1.3}$$

$$u_t + 3u^2 u_x + u_{xxx} = 0, \tag{1.4}$$

respectively, in a convenient sense. Results of orbital stability for equations (1.3) and (1.4) has been studied by many researchers in the case of solitary waves, for example see [1], [2], [3], [5], [6], [13] and [18]. Moreover, Kawahara equation is a model for small-amplitude gravity-capillary waves on water of a finite depth when the Weber number is close to $\frac{1}{3}$ (for details, see [16]). In this case, we have a break down when the Weber number is close to $\frac{1}{3}$. If the Weber number is larger than $\frac{1}{3}$, this equation has solitary waves just as the KdV approximation (see [9]).

Regarding the stability of solitary waves solutions, we can mention some contributors. In fact, Angulo

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in [4] showed the instability of solitary traveling-wave solutions associated with the generalized fifth-order KdV equation of the form

$$u_t + u_{xxxxx} + bu_{xxx} = (G(u, u_x, u_{xx}))_x, \quad (1.5)$$

where $G(q, r, s) = F_q(q, r) - rF_{qr}(q, r) - sF_{rr}(q, r)$ for some $F(q, r)$ which is homogeneous of degree $p+1$ for some $p > 1$, but the solitary wave was obtained by solving a constrained minimization problem in $H^2(\mathbb{R})$ which is based on results obtained by Levandosky (see [15]). The instability of this class of solitary-wave solutions is determined for $b \neq 0$, and it is obtained by making use of the variational characterization of the solitary waves and a modification of the theories of instability established by Shatah&Strauss [17], Bona&Souganidis&Strauss [6] and Gonçalves Ribeiro [12]. Levandosky's method in [15], was also used by Bridges&Derks [8] to show a result of the linear instability of solitary waves associated with the equation (1.5). However, the authors make use of a geometric approach.

We recall, from the results of Albert in [1], the solitary wave

$$u(x, t) = \varphi(x - c_0 t) = \text{sech}^4 \left(x - \frac{12}{35} t \right) \quad (1.6)$$

where $c_0 = \frac{12}{35}$, is a stable solution of the Kawahara equation,

$$u_t + uu_x + \frac{13}{420}u_{xxx} - \frac{1}{1680}u_{xxxxx} = 0.$$

In this result, the author used the nontrivial polynomials of Gegenbauer to determine the sign (strictly negative) of the quantity $I = (\chi, \varphi)_{L^2(\mathbb{R})}$. Here, $\chi \in L^2(\mathbb{R})$ is such that $\mathcal{L}\chi = \varphi$ (see Theorem 3.1 in Section 3).

Now, for more general dispersive evolution equations of the general form

$$u_t + u^p u_x - Mu_x = 0, \quad (1.7)$$

an important study of sufficient conditions for the stability was established by Albert in [1] (see also [2]) about solitary traveling waves of the form $u(x, t) = \varphi(x - ct)$, for the equation

$$(\mathcal{M} + c)\varphi - \frac{1}{p+1}\varphi^{p+1} = 0. \quad (1.8)$$

In (1.7) (and consequently in (1.8)), $p \geq 1$ is an integer and \mathcal{M} is a Fourier multiplier operator defined by

$$\widehat{\mathcal{M}g}(k) = \delta(k)\widehat{g}(k), \quad k \in \mathbb{R}, \quad (1.9)$$

where the symbol δ is a measurable, locally bounded, even function on \mathbb{R} and satisfies that $A_1|k|^\nu \leq \delta(k) \leq A_2(1 + |k|)^\mu$ for $\nu \leq \mu$, $|k| \geq k_0$, $\delta(k) > b$ for all $k \in \mathbb{R}$ and $A_i > 0$. In [1] sufficient conditions were determined to obtain that the linear, closed, unbounded, self-adjoint operator $\mathcal{L} : D(\mathcal{L}) \rightarrow L^2(\mathbb{R})$, defined on a dense subspace of $L^2(\mathbb{R})$ by

$$\mathcal{L}\zeta = (\mathcal{M} + c)\zeta - \varphi^p \zeta \quad (1.10)$$

where $\mathcal{M} + c$ is a positive operator, it will have exactly one negative eigenvalue which is simple and zero is simple with eigenfunction $\frac{d}{dx}\varphi$. These specific spectral properties of \mathcal{L} were obtained provided φ is a positive solitary wave satisfying that $\widehat{\varphi} > 0$ and $\widehat{\varphi^p} \in PF(2)$ class defined by Karlin in [14].

In this work, we will show two new explicit families of stable solitary traveling-wave solutions for the Kawahara and modified Kawahara equations (1.1) and (1.2) respectively. Such solitary waves are given, in the case of the Kawahara equation, by

$$\varphi_\omega(\xi) = \beta_1 \text{sech}^2(b\xi) + \lambda_1 \text{sech}^4(b\xi), \quad (1.11)$$

where $\omega > 0$ is the wave-speed and $\beta_1, \lambda_1, b > 0$ depending smoothly of ω . In the case of the modified Kawahara, we have

$$\phi_c(\xi) = \beta_2 \text{sech}^2(\alpha \xi) \quad (1.12)$$

where $c > 0$ is the wave speed with α and $\beta_2 > 0$ are parameters which depends smoothly of the wave-speed ω . However, in this specific case, we cannot obtain the nontrivial solitary traveling-wave solution associated with the modified Korteweg-de Vries equation (1.4) as $\gamma_2 \rightarrow 0$, namely the solitary traveling-wave solution $g_\omega(x) = 3\omega \text{sech}\left(\frac{\sqrt{\omega}x}{2}\right)$ associated with the equation (1.4). Note that in (1.11), if $\lambda_1 \rightarrow 0$ then, we could expect a profile solitary wave associated with the KdV equation (1.3).

For both cases, we will use the following conditions that imply the stability (see [5], [6], [13], and [18]):

- (P₀) there is a non-trivial smooth curve of solutions for (1.8) of the form,
 $c \in I \subseteq \mathbb{R} \rightarrow \varphi_c \in H^2(\mathbb{R});$
 - (P₁) \mathcal{L} has a unique negative eigenvalue λ , and which is simple;
 - (P₂) the eigenvalue 0 is simple;
 - (P₃) $\frac{d}{dc} \int_{\mathbb{R}} \varphi_c^2(x) dx > 0.$
- (1.13)

Therefore, by using conditions in (1.13) we are capable to investigate the nonlinear stability of the traveling-wave solutions of the forms (1.11) and (1.12) for the Kawahara (1.1) and modified Kawahara (1.2) equations, by using the theory developed by Albert [1] (see also Albert *et al.* [2]). Our stability result is derived from the ideas of Benjamin&Bona&Weinstein&Grillakis&Shatah&Strauss (see [5], [6], [13] and [18]).

In order to show the current findings, the paper is organized as follows. Section 2 establishes the notation used in the body of the paper and well-posedness results for the Kawahara and modified Kawahara equations. In Section 3 we present the general theory of stability and the main facts about the paper written by Albert [1]. Section 4 will show the existence of a branch of solitary traveling waves for the Kawahara equation and the respective proof of the stability. In Section 5 a branch of solitary traveling waves will be presented for the equation (1.2) and the respective proof of the stability for this case.

2. PRELIMINARIES AND WELL-POSEDNESS RESULTS.

We denote by \widehat{f} the Fourier transform of f in \mathbb{R} , which is defined as $\widehat{f}(\xi) = \int_{-\infty}^{\infty} f(x) e^{-i\xi x} dx$. Symbol $|f|_{L^p}$ denotes the $L^p(\mathbb{R})$ norm of f , $1 \leq p \leq \infty$. In particular, $|\cdot|_{L^2} = \|\cdot\|$ and $|\cdot|_{L^\infty} = \|\cdot\|_\infty$. The inner product of two elements $f, g \in L^2(\mathbb{R})$ will be denoted by $\langle f, g \rangle$. We denote by $H^s(\mathbb{R})$, $s \in \mathbb{R}$, the Sobolev space of all f (tempered distributions) for which the norm $\|f\|_{H^s}^2 = \int_{-\infty}^{\infty} (1 + |\xi|^2)^s |\widehat{f}(\xi)|^2 d\xi$ is finite.

2.1. Well-Posedness Results. An interesting result of well-posedness of the Kawahara equation in $L^2(\mathbb{R})$ is given by Cui&Deng&Tao in [10]. In the case of the modified Kawahara the result of well-posedness is given by Cui&Tao [11]. For both cases, the authors make use of the techniques of the Bourgain's spaces. These results can be summarized by the followings Theorems,

Theorem 2.1. *Let $s \geq 0$. For each $u_0 \in H^s(\mathbb{R})$ there is a $T > 0$ and a unique solution $u \in C([0, T]; H^s(\mathbb{R}))$ of the Kawahara equation (1.1). Moreover, the correspondence $u_0 \mapsto u$ is a continuous function between the adequate spaces*

Proof: See [10].

□

Theorem 2.2. *Let $s \geq 2$. For each $u_0 \in H^s(\mathbb{R})$ there is a $T > 0$ and a unique solution $u \in C([0, T]; H^s(\mathbb{R}))$ of the modified Kawahara equation (1.2). Moreover, the correspondence $u_0 \mapsto u$ is a continuous function between the adequate spaces*

Proof: See [11]. □

3. STABILITY THEOREM AND POSITIVITY PROPERTIES.

We start with our definition of stability

Definition 3.1. *Let φ be a solitary traveling-wave solution of the equation (1.1) (respectively 1.2) and consider $\tau_r \varphi(x) = \varphi(x + r)$, $x \in \mathbb{R}$ and $r \in \mathbb{R}$. We define the set $\Omega_\varphi \subset H^2(\mathbb{R})$, called the orbit generated by φ , as*

$$\Omega_\varphi = \{g; g = \tau_r \varphi, \text{ for some } r \in \mathbb{R}\}.$$

And for any $\eta > 0$, define the set $U_\eta \subset H^2(\mathbb{R})$ by

$$U_\eta = \left\{ f; \inf_{g \in \Omega_\varphi} \|f - g\|_{H^2} < \eta \right\}.$$

With this terminology, we say that φ is (orbitally) stable in $H^2(\mathbb{R})$ by the flow generated by equation (1.1) (respectively (1.2)) if,

- (i) the initial value problem associated with (1.1) (respectively (1.2)) is globally well-posed in $H^2(\mathbb{R})$ (see Theorems 2.1 and 2.2).
- (ii) For every $\varepsilon > 0$, there is $\delta > 0$ such that for all $u_0 \in U_\delta$, the solution u of (1.1) (respectively (1.2)) with $u(0, x) = u_0(x)$ satisfies $u(t) \in U_\varepsilon$ for all $t > 0$.

The proof of the following general stability Theorem can be obtained by using the techniques given by Benjamin [5], Bona [6], Weinstein [18] and Grillakis *et al.* [13].

Theorem 3.1. *Let φ be a solitary traveling-wave solution of (1.8) and suppose that part (i) of the definition of stability holds. Suppose also that the operator proceeding of the equation (1.8),*

$$\mathcal{L}\zeta = (\mathcal{M} + c)\zeta - \varphi^p \zeta, \tag{3.14}$$

determines that \mathcal{L} has exactly a unique negative eigenvalue which is simple and zero is a simple eigenvalue with eigenfunction $\frac{d}{dx}\varphi_c$. Choose $\chi \in L^2(\mathbb{R})$ such that $\mathcal{L}\chi = \varphi$ and define $I = (\chi, \varphi)_2$. If $I < 0$, then φ is stable. □

Remark 3.1. (i) *If condition (P_0) in (1.13) holds, we have in our case that function χ will be defined as $\chi = -\frac{d}{d\omega}\varphi_\omega$ or $\chi = -\frac{d}{dc}\phi_c$. Then, it is necessary to verify that $\frac{d}{d\omega}\|\varphi_\omega\|^2 > 0$ or $\frac{d}{dc}\|\phi_c\|^2 > 0$.*
(ii) *The existence of eigenvalues (and as consequence, eigenfunctions) for the operator (3.14) is guaranteed from the results contained in [3].*

The main result of the paper in Albert [1] (see also [2]) will be presented as follows. Before this, we need a preliminary definition

Definition 3.2. *We say that a function $g : \mathbb{R} \rightarrow \mathbb{R}$ is in the class $PF(2)$ if*

- i) $g(x) > 0$, $\forall x \in \mathbb{R}$,
- ii) $g(x_1 - x_2)g(x_2 - x_2) - g(x_1 - x_2)g(x_2 - x_1) > 0$ for $x_1 < x_2$ and $x_1 < x_2$.

Theorem 3.2. *Let φ be an even positive solution of (1.8). Suppose that $\widehat{\varphi} > 0$ and $\mathcal{K} = \widehat{\varphi^p} \in PF(2)$ discrete, then \mathcal{L} in (1.10) has exactly a unique negative eigenvalue which is simple and zero is a simple eigenvalue with eigenfunction $\frac{d}{dx}\varphi$.*

□

4. EXISTENCE AND STABILITY OF SOLITARY TRAVELING-WAVE SOLUTIONS FOR THE KAWAHARA EQUATION.

In this section we are interested in applying the theory developed by Albert in [1] to obtain the stability of a specific branch of positive solitary traveling waves associated with the Kawahara equation whose statements was presented in the previous section.

4.1. Existence of Solitary Traveling-Wave Solutions. In this subsection we establish the existence of solitary traveling-wave solutions related to the Kawahara equation given by,

$$u_t + uu_x + u_{xxx} - \gamma_1 u_{xxxx} = 0. \quad (4.15)$$

In fact, let $u(x, t) = \varphi_\omega(x - \omega t)$ be a solitary traveling-wave solution associated with (4.15). Substituting this form in the equation (4.15) we obtain, after integration, that

$$-\omega\varphi_\omega + \frac{1}{2}\varphi_\omega^2 + \varphi_\omega'' - \gamma_1\varphi_\omega'''' = 0, \quad (4.16)$$

where $\omega \in \mathbb{R}$.

Next, we consider

$$\varphi_\omega(\xi) = \beta_1 \text{sech}^2(b\xi) + \lambda_1 \text{sech}^4(b\xi), \quad (4.17)$$

where β_1, λ_1 and $b > 0$ a smooth solution for (4.16). By using *Maple program*, the following nonlinear system is obtained,

$$\begin{cases} \lambda_1 - 1680b^4\gamma_1 = 0 \\ -\frac{1}{2}b^2 + \gamma_1 b^4 + \frac{1}{16}\omega = 0 \\ 240\gamma_1 b^4\beta_1 - 512\gamma_1 b^4\lambda_1 - 32\lambda_1 b^2 - 12\beta_1 b^2 + \beta_1^2 - 2\omega\lambda_1 = 0 \\ 2080\gamma_1 b^4\lambda_1 - 240\gamma_1\beta_1 b^4 - 40\lambda_1 b^2 + 2\beta_1\lambda_1 = 0. \end{cases} \quad (4.18)$$

After some calculations, (4.18) boils down in a simple system as

$$\begin{cases} b^2 - \frac{\lambda_1}{840} - \frac{\omega}{8} = 0 \\ 26\lambda_1 + 39\beta_1 - 840b^2 = 0 \\ 3\lambda_1\beta_1 - 32\lambda_1^2 - 672\lambda_1 b^2 - 252\beta_1 b^2 + 21\beta_1^2 - 42\omega\lambda_1 = 0 \end{cases} \quad (4.19)$$

System (4.19) can be dropped in terms of λ_1 and ω as

$$-\frac{2023210}{169}\omega\lambda_1 - \frac{862463}{507}\lambda_1^2 + \frac{797475}{169}\omega^2 = 0. \quad (4.20)$$

Then, we discover λ_1 in terms of ω

$$\lambda_1(\omega) = 105 \left(-\frac{4129}{123209} + \frac{546}{123209} \sqrt{70} \right) \omega. \quad (4.21)$$

where we can conclude that $\lambda_1(\omega) > 0$ and $\lambda_1'(\omega) > 0$ for all $\omega > 0$.

Further, from (4.19) and (4.21) we have

$$\begin{aligned} \beta_1(\omega) &= \frac{105\omega}{39} - \frac{25\lambda_1}{39} \\ &= \frac{(609630 - 36750\sqrt{70})\omega}{123209}. \end{aligned} \quad (4.22)$$

Therefore, we get $\beta_1(\omega) > 0$ and $\beta_1'(\omega) > 0$ for all $\omega > 0$.

Finally, we can find b in term of ω as

$$b(\omega) = \frac{\sqrt{123209} \sqrt{(59540 + 273\sqrt{70})\omega}}{246418} \quad (4.23)$$

and we have $b(\omega) > 0$ and $b'(\omega) > 0$ for all $\omega > 0$.

Next, from (4.21), (4.22) and (4.23) we deduce that

$$\omega \in (0, +\infty) \mapsto \varphi_\omega \in H^n(\mathbb{R}) \text{ is smooth for all } n \in \mathbb{N}. \quad (4.24)$$

4.2. Stability of Solitary Traveling-Wave Solutions. We have the following Theorem of stability

Theorem 4.1. *The smooth branch of solutions φ_ω obtained in (4.24) is orbitally stable in $H^2(\mathbb{R})$ by the flow of the Kawahara equation since $\omega > 0$.*

Proof: First of all we wish to determine the behavior of the first two eigenvalues associated with the operator

$$\mathcal{L} = \gamma_1 \frac{d^4}{dx^4} - \frac{d^2}{dx^2} + \omega - \varphi_\omega,$$

by utilizing the theory developed by Albert in [1]. In fact, since the kernel $\mathcal{K} = \widehat{\varphi_\omega}$ belongs to the $PF(2)$ continuous case from the Lemma 10 in [2], it is necessary to show that the symbol $\delta(z)$ associated with the linear operator $\mathcal{M} = \gamma_1 \frac{d^4}{dx^4} - \frac{d^2}{dx^2}$ satisfies the properties in (1.9). Indeed, since $\delta(z) = \widehat{Mu}(z) = (\gamma_1 |z|^4 + |z|^2) \widehat{u}(z)$ for all $z \in \mathbb{R}$, the properties are verified.

Next, from Theorem 3.1 and Remark 3.1-(i) we calculate $\frac{d}{d\omega} \|\varphi_\omega\|^2$, where $\varphi_\omega(\xi) = \beta_1 \operatorname{sech}^2(b\xi) + \lambda_1 \operatorname{sech}^4(b\xi)$ and $\omega > 0$. In fact

$$\begin{aligned} \|\varphi_\omega\|^2 &= \frac{\beta_1^2}{b} \int_{\mathbb{R}} \operatorname{sech}^4(x) dx + \frac{2\beta_1\lambda_1}{b} \int_{\mathbb{R}} \operatorname{sech}^6(x) dx + \frac{\lambda_1^2}{b} \int_{\mathbb{R}} \operatorname{sech}^8(x) dx \\ &= \frac{4\beta_1^2}{3b} + \frac{32\beta_1\lambda_1}{15b} + \frac{32\lambda_1^2}{35b} \end{aligned}$$

Since $\frac{\beta_1^2}{b} = M_1 \omega^{3/2}$, $\frac{\beta_1\lambda_1}{b} = M_2 \omega^{3/2}$, $\frac{\lambda_1^2}{b} = M_3 \omega^{3/2}$, where M_i , $i = 1, 2, 3$ are positive constants obtained from (4.21), (4.22) and (4.23) we deduce that $d''(\omega) > 0$, for all $\omega > 0$.

□

5. EXISTENCE AND STABILITY OF SOLITARY TRAVELING-WAVE SOLUTIONS FOR THE MODIFIED KAWAHARA EQUATION.

This section is concerned to proof the existence and stability of solitary traveling-wave solutions for the modified Kawahara equation

$$-c\phi_c + \phi_c^3 + \phi_c'' - \gamma_2\phi_c'''' = 0, \quad (5.25)$$

where $\gamma_2 > 0$.

To prove the existence, let $\phi_c(\xi) = \beta_2 \text{sech}^2(\alpha\xi)$ be a solitary traveling-wave solution for the equation (5.25). If we substitute this ϕ_c into (5.25) we obtain after some calculations $\beta_2 = 6\alpha$ and $\alpha = \frac{\sqrt{5c}}{4}$. Therefore for all $c > 0$ we have,

$$\phi_c(\xi) = \frac{3\sqrt{5c}}{2} \text{sech}^2\left(\frac{\sqrt{5c}}{4}\xi\right). \quad (5.26)$$

5.1. Stability of Solitary Traveling-Wave Solutions. In this subsection we are interested in applying the theory in Section 3 to obtain the stability of the smooth branch of positive solitary traveling waves obtained in the last subsection, associated to the modified Kawahara equation. In fact, our intention can be summarized in the following theorem,

Theorem 5.1. *The branch of solutions ϕ_c given by (5.26) is orbitally stable in $H^2(\mathbb{R})$ by the flow of the modified Kawahara equation for all $c > 0$.*

Proof: First of all, we note clearly that $\widehat{\phi_c} > 0$. Lemma 10 in [2], also shows that $\widehat{\phi_c^2}$ belongs to the $PF(2)$ class and therefore the properties (P_1) and (P_2) in (1.13) are satisfied for the operator $\mathcal{L} = \gamma_2 \frac{d^4}{dx^4} - \frac{d^2}{dx^2} + c - 3\phi_c^2$, associated with the equation (5.25). Moreover, the linear operator $\mathcal{M} = \gamma_2 \frac{d^4}{dx^4} - \frac{d^2}{dx^2}$, satisfies the properties required in (1.9) by the same arguments seen in the proof of the Theorem 4.1. It remains for us to calculate the quantity $\frac{d}{dc} \|\phi_c\|^2$. In fact, since

$$\int_{\mathbb{R}} \phi_c(\xi)^2 d\xi = \frac{45\sqrt{c}}{\sqrt{5}} \int_{\mathbb{R}} \text{sech}^4(y) dy = \frac{60\sqrt{c}}{\sqrt{5}},$$

we have $\frac{d}{dc} \|\phi_c\|^2 > 0$. This argument shows the theorem. □

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